

# Time-dependent Neutral stochastic functional differential equation driven by a fractional Brownian motion in a Hilbert space

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## Abstract

In this paper we consider a class of time-dependent neutral stochastic functional differential equations with finite delay driven by a fractional Brownian motion in a Hilbert space. We prove an existence and uniqueness result for the mild solution by means of the Banach fixed point principle. A practical example is provided to illustrate the viability of the abstract result of this work.

*Keywords:* Neutral stochastic evolution equation. Evolution operator.

Fractional Brownian motion. Wiener integral. Banach fixed point theorem.

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## 1. Introduction

The stochastic functional differential equations have attracted much attention because of their practical applications in many areas such as physics, medicine, biology, finance, population dynamics, electrical engineering, telecommunication networks, and other fields. For more details, one can see Da Prato and Zabczyk (1992), and Ren and Sun (2009) and the references therein.

In many areas of science, there has been an increasing interest in the investigation of the systems incorporating memory or aftereffect, i.e., there is the effect of delay on state equations. Therefore, there is a real need to discuss stochastic

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evolution systems with delay. In many mathematical models the claims often display long-range memories, possibly due to extreme weather, natural disasters, in some cases, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays. In such cases, the class of neutral stochastic differential equations driven by fractional Brownian motion provide an important tool for describing and analyzing such systems. Very recently, neutral stochastic functional differential equations driven by fractional Brownian motion have attracted the interest of many researchers. One can see Boufoussi and Hajji (2012), Caraballo et al. (2011), Hajji and Lakhel (2013) and the references therein.

Motivated by the above works, this paper is concerned with the existence and uniqueness of mild solutions for a class of time-dependent neutral functional stochastic differential equations described in the form:

$$\begin{cases} d[x(t) + g(t, x(t - r(t)))] = [A(t)x(t) + f(t, x(t - \rho(t)))]dt + \sigma(t)dB^H(t), & 0 \leq t \leq T, \\ x(t) = \varphi(t), & -\tau \leq t \leq 0, \end{cases} \quad (1)$$

where  $\{A(t), t \in [0, T]\}$  is a family of linear closed operators from a Hilbert space  $X$  into  $X$  that generates an evolution system of operators  $\{U(t, s), 0 \leq s \leq t \leq T\}$ .  $B^H$  is a fractional Brownian motion on a real and separable Hilbert space  $Y$ ,  $r, \rho : [0, +\infty) \rightarrow [0, \tau]$  ( $\tau > 0$ ) are continuous and  $f, g : [0, +\infty) \times X \rightarrow X$ ,  $\sigma : [0, +\infty) \rightarrow \mathcal{L}_2^0(Y, X)$ , are appropriate functions. Here  $\mathcal{L}_2^0(Y, X)$  denotes the space of all  $Q$ -Hilbert-Schmidt operators from  $Y$  into  $X$  (see section 2 below).

On the other hand, to the best of our knowledge, there is no paper which investigates the study of time-dependent neutral stochastic functional differential equations with delays driven by fractional Brownian motion. Thus, we will make the first attempt to study such problem in this paper.

Our results are inspired by the one in Boufoussi and Hajji (2012) where the existence and uniqueness of mild solutions to model (1) with  $A(t) = A, \forall t \in [0, T]$ , is studied, as well as some results on the asymptotic behavior.

The substance of the paper is organized as follows. Section 2, recapitulate some notations, basic concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and we recall some preliminary results about evolution operator. We need to prove a new technical lemma for the  $\mathbb{L}^2$ -estimate of stochastic convolution integral which is different from that used by Boufoussi and Hajji (2012). Section 3, gives sufficient conditions to prove the existence and uniqueness for the problem (1). In Section 4 we give an example to illustrate the efficiency of the obtained result.

## 2. Preliminaries

### 2.1. Evolution families

In this subsection we introduce the notion of evolution family.

**Definition 1.** A set  $\{U(t, s) : 0 \leq s \leq t \leq T\}$  of bounded linear operators on a Hilbert space  $X$  is called an evolution family if

- (a)  $U(t, s)U(s, r) = U(t, r)$ ,  $U(s, s) = I$  if  $r \leq s \leq t$ ,
- (b)  $(t, s) \rightarrow U(t, s)x$  is strongly continuous for  $t > s$ .

Let  $\{A(t), t \in [0, T]\}$  be a family of closed densely defined linear unbounded operators on the Hilbert space  $X$  and with domain  $D(A(t))$  independent of  $t$ , satisfying the following conditions introduced by Acquistapace and Terreni (1987).

There exist constants  $\lambda_0 \geq 0$ ,  $\theta \in (\frac{\pi}{2}, \pi)$ ,  $L, K \geq 0$ , and  $\mu, \nu \in (0, 1]$  with  $\mu + \nu > 1$  such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|} \quad (2)$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t - s|^\mu |\lambda|^{-\nu}, \quad (3)$$

for  $t, s \in \mathbb{R}$ ,  $\lambda \in \Sigma_\theta$  where  $\Sigma_\theta := \{\lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \leq \theta\}$ .

It is well known, that this assumption implies that there exists a unique evolution family  $\{U(t, s) : 0 \leq s \leq t \leq T\}$  on  $X$  such that  $(t, s) \rightarrow U(t, s) \in \mathcal{L}(X)$  is continuous for  $t > s$ ,  $U(\cdot, s) \in \mathcal{C}^1((s, \infty), \mathcal{L}(X))$ ,  $\partial_t U(t, s) = A(t)U(t, s)$ , and

$$\|A(t)^k U(t, s)\| \leq C(t - s)^{-k} \quad (4)$$

for  $0 < t - s \leq 1$ ,  $k = 0, 1$ ,  $0 \leq \alpha < \mu$ ,  $x \in D((\lambda_0 - A(s))^\alpha)$ , and a constant  $C$  depending only on the constants in (2)-(3). Moreover,  $\partial_s^+ U(t, s)x = -U(t, s)A(s)x$  for  $t > s$  and  $x \in D(A(s))$  with  $A(s)x \in \overline{D(A(s))}$ . We say that  $A(\cdot)$  generates  $\{U(t, s) : 0 \leq s \leq t \leq T\}$ . Note that  $U(t, s)$  is exponentially bounded by (4) with  $k = 0$ .

**Remark 2.** *If  $\{A(t), t \in [0, T]\}$  is a second order differential operator  $A$ , that is  $A(t) = A$  for each  $t \in [0, T]$ , then  $A$  generates a  $C_0$ -semigroup  $\{e^{At}, t \in [0, T]\}$ .*

For additional details on evolution system and their properties, we refer the reader to Pazy (1983).

## 2.2. Fractional Brownian Motion

For the convenience for the reader we recall briefly here some of the basic results of fractional Brownian motion calculus. For details of this section, we refer the reader to Nualart (2006) and the references therein.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space. A standard fractional Brownian motion (fBm)  $\{\beta^H(t), t \in \mathbb{R}\}$  with Hurst parameter  $H \in (0, 1)$  is a zero mean Gaussian process with continuous sample paths such that

$$\mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) \quad (5)$$

for  $s, t \in \mathbb{R}$ . It is clear that for  $H = 1/2$ , this process is a standard Brownian motion. In this paper, it is assumed that  $H \in (\frac{1}{2}, 1)$ .

This process was introduced by Kolmogorov (1940) and later studied by Mandelbrot and Van Ness (1968). Its self-similar and long-range dependence

make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Consider a time interval  $[0, T]$  with arbitrary fixed horizon  $T$  and let  $\{\beta^H(t), t \in [0, T]\}$  the one-dimensional fractional Brownian motion with Hurst parameter  $H \in (1/2, 1)$ . It is well known that  $\beta^H$  has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) d\beta(s), \quad (6)$$

where  $\beta = \{\beta(t) : t \in [0, T]\}$  is a Wiener process, and  $K_H(t; s)$  is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for  $t > s$ , where  $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$  and  $\beta(\cdot, \cdot)$  denotes the Beta function. We put  $K_H(t, s) = 0$  if  $t \leq s$ .

We will denote by  $\mathcal{H}$  the reproducing kernel Hilbert space of the fBm. In fact  $\mathcal{H}$  is the closure of the set of indicator functions  $\{1_{[0,t]}, t \in [0, T]\}$  with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping  $1_{[0,t]} \rightarrow \beta^H(t)$  can be extended to an isometry between  $\mathcal{H}$  and the first Wiener chaos and we will denote by  $\beta^H(\varphi)$  the image of  $\varphi$  by the previous isometry.

We recall that for  $\psi, \varphi \in \mathcal{H}$  their scalar product in  $\mathcal{H}$  is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s) \varphi(t) |t-s|^{2H-2} ds dt.$$

Let us consider the operator  $K_H^*$  from  $\mathcal{H}$  to  $\mathbb{L}^2([0, T])$  defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r, s) dr.$$

We refer to Nualart (2006) for the proof of the fact that  $K_H^*$  is an isometry between  $\mathcal{H}$  and  $L^2([0, T])$ . Moreover for any  $\varphi \in \mathcal{H}$ , we have

$$\beta^H(\varphi) = \int_0^T (K_H^* \varphi)(t) d\beta(t).$$

It follows from Nualart (2006) that the elements of  $\mathcal{H}$  may be not functions but distributions of negative order. In the case  $H > \frac{1}{2}$ , the second partial derivative of the covariance function

$$\frac{\partial R_H}{\partial t \partial s} = \alpha_H |t - s|^{2H-2},$$

where  $\alpha_H = H(2H - 2)$ , is integrable, and we can write

$$R_H(t, s) = \alpha_H \int_0^t \int_0^s |u - v|^{2H-2} du dv. \quad (7)$$

In order to obtain a space of functions contained in  $\mathcal{H}$ , we consider the linear space  $|\mathcal{H}|$  generated by the measurable functions  $\psi$  such that

$$\|\psi\|_{|\mathcal{H}|}^2 := \alpha_H \int_0^T \int_0^T |\psi(s)| |\psi(t)| |s - t|^{2H-2} ds dt < \infty,$$

where  $\alpha_H = H(2H - 1)$ . The space  $|\mathcal{H}|$  is a Banach space with the norm  $\|\psi\|_{|\mathcal{H}|}$  and we have the following inclusions (see Nualart (2006)).

**Lemma 3.**

$$\mathbb{L}^2([0, T]) \subseteq \mathbb{L}^{1/H}([0, T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any  $\varphi \in \mathbb{L}^2([0, T])$ , we have

$$\|\varphi\|_{|\mathcal{H}|}^2 \leq 2HT^{2H-1} \int_0^T |\varphi(s)|^2 ds.$$

Let  $X$  and  $Y$  be two real, separable Hilbert spaces and let  $\mathcal{L}(Y, X)$  be the space of bounded linear operator from  $Y$  to  $X$ . For the sake of convenience, we shall use the same notation to denote the norms in  $X, Y$  and  $\mathcal{L}(Y, X)$ . Let  $Q \in \mathcal{L}(Y, Y)$  be an operator defined by  $Qe_n = \lambda_n e_n$  with finite trace  $\text{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty$ . where  $\lambda_n \geq 0$  ( $n = 1, 2, \dots$ ) are non-negative real numbers and  $\{e_n\}$  ( $n = 1, 2, \dots$ ) is a complete orthonormal basis in  $Y$ . Let  $B^H = (B^H(t))$  be  $Y$ -valued fBm on  $(\Omega, \mathcal{F}, \mathbb{P})$  with covariance  $Q$  as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where  $\beta_n^H$  are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x \rangle \langle B^H(s), y \rangle = R(s, t) \langle Q(x), y \rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0, T]$$

In order to define Wiener integrals with respect to the  $Q$ -fBm, we introduce the space  $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$  of all  $Q$ -Hilbert-Schmidt operators  $\psi : Y \rightarrow X$ . We recall that  $\psi \in \mathcal{L}(Y, X)$  is called a  $Q$ -Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty,$$

and that the space  $\mathcal{L}_2^0$  equipped with the inner product  $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$  is a separable Hilbert space.

Now, let  $\phi(s); s \in [0, T]$  be a function with values in  $\mathcal{L}_2^0(Y, X)$ , such that  $\sum_{n=1}^{\infty} \|K^* \phi Q^{\frac{1}{2}} e_n\|_{\mathcal{L}_2^0}^2 < \infty$ . The Wiener integral of  $\phi$  with respect to  $B^H$  is defined by

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} (K_H^* (\phi e_n))(s) d\beta_n(s) \quad (8)$$

where  $\beta_n$  is the standard Brownian motion used to present  $\beta_n^H$  as in (6).

Now, we end this subsection by stating the following result which is fundamental to prove our result. It can be proved by similar arguments as those used to prove Lemma 2 in Caraballo et al. (2011).

**Lemma 4.** *If  $\psi : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$  satisfies  $\int_0^T \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$ , then the above sum in (8) is well defined as a  $X$ -valued random variable and we have*

$$\mathbb{E} \left\| \int_0^t \psi(s) dB^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

### 2.3. The stochastic convolution integral

In this subsection we present a few properties of the stochastic convolution integral of the form

$$Z(t) = \int_0^t U(t, s) \sigma(s) dB^H(s), \quad t \in [0, T],$$

where  $\sigma(s) \in \mathcal{L}_2^0(Y, X)$  and  $\{U(t, s), 0 \leq s \leq t \leq T\}$  is an evolution system of operators.

The properties of the process  $Z$  are crucial when regularity of the mild solution to stochastic evolution equation is studied, see Da Prato and Zabczyk (1992) for asystematic account of the theory of mild solutions to infinite-dimensional stochastic equations. Unfortunately, the process  $Z$  is not a martingale, and standard tools of the martingale theory, yielding e.g. continuity of the trajectories or  $\mathbb{L}^2$ -estimates are not available.

The following result on the stochastic convolution integral  $Z$  holds.

**Lemma 5.** *Suppose that  $\sigma : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$  satisfies  $\sup_{t \in [0, T]} \|\sigma(t)\|_{\mathcal{L}_2^0}^2 < \infty$ , and Suppose that  $\{U(t, s), 0 \leq s \leq t \leq T\}$  is an evolution system of operators satisfying  $\|U(t, s)\| \leq Me^{-\beta(t-s)}$ , for some constants  $\beta > 0$  and  $M \geq 1$  for all  $t \geq s$ . Then, we have*

$$\mathbb{E} \left\| \int_0^t U(t, s) \sigma(s) dB^H(s) \right\|^2 \leq CM^2 t^{2H} \left( \sup_{t \in [0, T]} \|\sigma(t)\|_{\mathcal{L}_2^0} \right)^2.$$

**Proof.** Let  $\{e_n\}_{n \in \mathbb{N}}$  be the complete orthonormal basis of  $Y$  and  $\{\beta_n^H\}_{n \in \mathbb{N}}$  is a sequence of independent, real-valued standard fractional Brownian motion each with the same Hurst parameter  $H \in (\frac{1}{2}, 1)$ . Thus, using fractional Itô isometry one can write

$$\begin{aligned} \mathbb{E} \left\| \int_0^t U(t, s) \sigma(s) dB^H(s) \right\|^2 &= \sum_{n=1}^{\infty} \mathbb{E} \left\| \int_0^t U(t, s) \sigma(s) e_n d\beta_n^H(s) \right\|^2 \\ &= \sum_{n=1}^{\infty} \int_0^t \int_0^t \langle U(t, s) \sigma(s) e_n, U(t, r) \sigma(r) e_n \rangle \\ &\quad \times H(2H-1) |s-r|^{2H-2} ds dr \\ &\leq H(2H-1) \int_0^t \{ \|U(t, s) \sigma(s)\| \\ &\quad \times \int_0^t \|U(t, r) \sigma(r)\| |s-r|^{2H-2} dr \} ds \\ &\leq H(2H-1) M^2 \int_0^t \{ e^{-\beta(t-s)} \|\sigma(s)\|_{\mathcal{L}_2^0} \\ &\quad \times \int_0^t e^{-\beta(t-r)} |s-r|^{2H-2} \|\sigma(r)\|_{\mathcal{L}_2^0} dr \} ds. \end{aligned}$$



Since  $\sigma$  is bounded, one can then conclude that

$$\begin{aligned} \mathbb{E} \left\| \int_0^t U(t, s) \sigma(s) dB^H(s) \right\|^2 &\leq H(2H-1)M^2 \left( \sup_{t \in [0, T]} \|\sigma(t)\|_{\mathcal{L}_2^0} \right)^2 \int_0^t \{e^{-\beta(t-s)} \\ &\quad \times \int_0^t e^{-\beta(t-r)} |s-r|^{2H-2} dr\} ds. \end{aligned}$$

Make the following change of variables,  $v = t - s$  for the first integral and  $u = t - r$  for the second. One can write

$$\begin{aligned} \mathbb{E} \left\| \int_0^t U(t, s) \sigma(s) dB^H(s) \right\|^2 &\leq H(2H-1)M^2 \left( \sup_{t \in [0, T]} \|\sigma(t)\|_{\mathcal{L}_2^0} \right)^2 \int_0^t \{e^{-\beta v} \\ &\quad \times \int_0^t e^{-\beta u} |u-v|^{2H-2} du\} dv \\ &\leq H(2H-1)M^2 \left( \sup_{t \in [0, T]} \|\sigma(t)\|_{\mathcal{L}_2^0} \right)^2 \int_0^t \int_0^t |u-v|^{2H-2} dudv. \end{aligned}$$

By using (7), we get that

$$\mathbb{E} \left\| \int_0^t U(t, s) \sigma(s) dB^H(s) \right\|^2 \leq CM^2 t^{2H} \left( \sup_{t \in [0, T]} \|\sigma(t)\|_{\mathcal{L}_2^0} \right)^2.$$

■

**Remark 6.** Thanks to Lemma 5, the stochastic integral  $Z(t)$  is well-defined.

### 3. Existence and Uniqueness of Mild Solutions

In this section we study the existence and uniqueness of mild solutions of equation (1). Henceforth we will assume that the family  $\{A(t), t \in [0, T]\}$  of linear operators generates an evolution system of operators  $\{U(t, s), 0 \leq s \leq t \leq T\}$ .

Before stating and proving the main result, we give the definition of mild solutions for equation (1).

**Definition 7.** A  $X$ -valued process  $\{x(t), t \in [-r, T]\}$ , is called a mild solution of equation (1) if

$$i) \ x(\cdot) \in \mathcal{C}([-r, T], \mathbb{L}^2(\Omega, X)),$$

$$ii) \ x(t) = \varphi(t), \ -r \leq t \leq 0.$$

iii) For arbitrary  $t \in [0, T]$ ,  $x(t)$  satisfies the following integral equation:

$$\begin{aligned} x(t) &= U(t, 0)(\varphi(0) + g(0, \varphi(-\rho(0)))) - g(t, x(t - \rho(t))) \\ &\quad - \int_0^t U(t, s)A(s)g(s, x(s - \rho(s)))ds + \int_0^t U(t, s)f(s, x(s - \rho(s)))ds \\ &\quad + \int_0^t U(t, s)\sigma(s)dB^H(s) \quad \mathbb{P} - a.s \end{aligned}$$

We introduce the following assumptions:

(H.1) i) The evolution family is exponentially stable, that is, there exist two constants  $\beta > 0$  and  $M \geq 1$  such that

$$\|U(t, s)\| \leq Me^{-\beta(t-s)}, \quad \text{for all } t \geq s,$$

ii) There exist a constant  $M_* > 0$  such that

$$\|A^{-1}(t)\| \leq M_* \quad \text{for all } t \in [0, T].$$

(H.2) The maps  $f, g : [0, T] \times X \rightarrow X$  are continuous functions and there exist two positive constants  $C_1$  and  $C_2$ , such that for all  $t \in [0, T]$  and  $x, y \in X$ :

$$\begin{aligned} i) \quad &\|f(t, x) - f(t, y)\| \vee \|g(t, x) - g(t, y)\| \leq C_1\|x - y\|. \\ ii) \quad &\|f(t, x)\|^2 \vee \|A^k(t)g(t, x)\|^2 \leq C_2(1 + \|x\|^2), \quad k = 0, 1. \end{aligned}$$

(H.3) i) There exists a constant  $0 < L_* < \frac{1}{M_*}$  such that

$$\|A(t)g(t, x) - A(t)g(t, y)\| \leq L_*\|x - y\|,$$

for all  $t \in [0, T]$  and  $x, y \in X$ .

ii) The function  $g$  is continuous in the quadratic mean sense: for all  $x(\cdot) \in \mathcal{C}([0, T], L^2(\Omega, X))$ , we have

$$\lim_{t \rightarrow s} \mathbb{E}\|g(t, x(t)) - g(s, x(s))\| = 0.$$

(H.4) i) The map  $\sigma : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$  is bounded, that is : there exists a positive constant such that  $\|\sigma(t)\|_{\mathcal{L}_2^0(Y, X)} \leq L$  uniformly in  $t \in [0, T]$ .

ii) The maps  $r, \rho : [0, \infty] \rightarrow \mathbb{R}$  are a continuous functions satisfying the condition that

$$-\tau \leq \rho(t), r(t) \leq t, \forall t \geq 0.$$

Moreover, we assume that  $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$ .

The main result of this paper is given in the next theorem.

**Theorem 8.** *Suppose that (H.1)-(H.4) hold. Then, for all  $T > 0$ , the equation (1) has a unique mild solution on  $[-\tau, T]$ .*

**Proof.**

Fix  $T > 0$  and let  $B_T := \mathcal{C}([-\tau, T], \mathbb{L}^2(\Omega, X))$  be the Banach space of all continuous functions from  $[-\tau, T]$  into  $\mathbb{L}^2(\Omega, X)$ , equipped with the supremum norm

$$\|x\|_{B_T}^2 = \sup_{-\tau \leq t \leq T} \mathbb{E} \|x(t, \omega)\|^2.$$

Let us consider the set

$$S_T(\varphi) = \{x \in B_T : x(s) = \varphi(s), \text{ for } s \in [-\tau, 0]\}.$$

$S_T(\varphi)$  is a closed subset of  $B_T$  provided with the norm  $\|\cdot\|_{B_T}$ .

We transform (1) into a fixed-point problem. Consider the operator  $\psi$  on  $S_T(\varphi)$  defined by  $\psi(x)(t) = \varphi(t)$  for  $t \in [-\tau, 0]$  and for  $t \in [0, T]$

$$\begin{aligned} \psi(x)(t) &= U(t, 0)(\varphi(0) + g(0, \varphi(r(0)))) - g(t, x(t - r(t))) \\ &\quad - \int_0^t U(t, s)A(s)g(s, x(s - r(s)))ds + \int_0^t U(t, s)f(s, x(s - \rho(s)))ds \\ &\quad + \int_0^t U(t, s)\sigma(s)dB^H(s) \\ &= \sum_{i=1}^5 I_i(t). \end{aligned}$$

Clearly, the fixed points of the operator  $\psi$  are mild solutions of (1). The fact that  $\psi$  has a fixed point will be proved in several steps. We will first prove that the function  $\psi$  is well defined.

**Step 1:**  $\psi$  is well defined. Let  $x \in S_T(\varphi)$  and  $t \in [0, T]$ , we are going to show

that each function  $t \rightarrow I_i(t)$  is continuous on  $[0, T]$  in the  $\mathbb{L}^2(\Omega, X)$ -sense.

From Definition 1, we obtain

$$\lim_{h \rightarrow 0} (U(t+h, 0) - U(t, 0))(\varphi(0) + g(0, \varphi(r(0)))) = 0.$$

From (H.1), we have

$$\|(U(t+h, 0) - U(t, 0))(\varphi(0) + g(0, \varphi(r(0))))\| \leq M e^{-\beta t} (e^{-\beta h} + 1) \|\varphi(0) + g(0, \varphi(r(0)))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h \rightarrow 0} \mathbb{E} \|I_1(t+h) - I_1(t)\|^2 = 0.$$

Moreover, assumption (H.2) ensures that

$$\lim_{h \rightarrow 0} \mathbb{E} \|I_2(t+h) - I_2(t)\|^2 = 0.$$

To show that the third term  $I_3(h)$  is continuous, we suppose  $h > 0$  (similar calculus for  $h < 0$ ). We have

$$\begin{aligned} |I_3(t+h) - I_3(t)| &\leq \left| \int_0^t (U(t+h, s) - U(t, s)) A(s) g(s, x(s-r(s))) ds \right| \\ &\quad + \left| \int_t^{t+h} (U(t, s) g(s, x(s-r(s)))) ds \right| \\ &\leq I_{31}(h) + I_{32}(h). \end{aligned}$$

By Hölder's inequality, we have

$$\mathbb{E} \|I_{41}(h)\| \leq t \mathbb{E} \int_0^t \|U(t+h, s) - U(t, s)\| A(s) g(s, x(s-r(s)))^2 ds.$$

By Definition 1, we obtain

$$\lim_{h \rightarrow 0} (U(t+h, s) - U(t, s)) A(s) g(s, x(s-r(s))) = 0.$$

From (H.1) and (H.2), we have

$$\|U(t+h, s) - U(t, s)\| A(s) g(s, x(s-r(s))) \leq C_2 M e^{-\beta(t-s)} (e^{-\beta h} + 1) \|A(s) g(s, x(s-r(s)))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h \rightarrow 0} \mathbb{E} \|I_{31}(h)\|^2 = 0.$$

So, estimating as before. By using  $(\mathcal{H}.1)$  and  $(\mathcal{H}.2)$ , we get

$$\mathbb{E} \|I_{32}(h)\|^2 \leq \frac{M^2 C_2 (1 - e^{-2\beta h})}{2\beta} \int_t^{t+h} (1 + \mathbb{E} \|x(s - r(s))\|^2) ds.$$

Thus,

$$\lim_{h \rightarrow 0} \mathbb{E} \|I_{32}(h)\|^2 = 0.$$

For the fourth term  $I_4(h)$ , we suppose  $h > 0$  (similar calculus for  $h < 0$ ).

We have

$$\begin{aligned} |I_4(t+h) - I_4(t)| &\leq \left| \int_0^t (U(t+h, s) - U(t, s)) f(s, x(s - \rho(s))) ds \right| \\ &\quad + \left| \int_t^{t+h} (U(t, s) f(s, x(s - \rho(s)))) ds \right| \\ &\leq I_{41}(h) + I_{42}(h). \end{aligned}$$

By Hölder's inequality, we have

$$\mathbb{E} \|I_{41}(h)\| \leq t \mathbb{E} \int_0^t \|U(t+h, s) - U(t, s)\| f(s, x(s - \rho(s)))^2 ds.$$

Again exploiting properties of Definition 1, we obtain

$$\lim_{h \rightarrow 0} (U(t+h, s) - U(t, s)) f(s, x(s - \rho(s))) = 0,$$

and

$$\|U(t+h, s) - U(t, s)\| f(s, x(s - \rho(s))) \leq M e^{-\beta(t-s)} (e^{-\beta h} + 1) \|f(s, x(s - \rho(s)))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h \rightarrow 0} \mathbb{E} \|I_{41}(h)\|^2 = 0.$$

On the other hand, by  $(\mathcal{H}.1)$ ,  $(\mathcal{H}.2)$ , and the Hölder's inequality, we have

$$\mathbb{E} \|I_{42}(h)\| \leq \frac{M^2 C_2 (1 - e^{-2\beta h})}{2\beta} \int_t^{t+h} (1 + \mathbb{E} \|x(s - \rho(s))\|^2) ds.$$

Thus

$$\lim_{h \rightarrow 0} I_{42}(h) = 0.$$

Now, for the term  $I_5(h)$ , we have

$$\begin{aligned} I_5(h) &\leq \left\| \int_0^t (U(t+h, s) - U(t, s)) \sigma(s) dB^H(s) \right\| \\ &\quad + \left\| \int_t^{t+h} U(t+h, s) \sigma(s) dB^H(s) \right\| \\ &\leq I_{51}(h) + I_{52}(h). \end{aligned}$$

By Lemma 4, we get that

$$E|I_{51}(h)|^2 \leq 2Ht^{2H-1} \int_0^t \| [U(t+h, s) - U(t, s)] \sigma(s) \|_{\mathcal{L}_2^0}^2 ds.$$

Since

$$\lim_{h \rightarrow 0} \| [U(t+h, s) - U(t, s)] \sigma(s) \|_{\mathcal{L}_2^0}^2 = 0$$

and

$$\| (U(t+h, s) - U(t, s)) \sigma(s) \|_{\mathcal{L}_2^0} \leq (ML e^{-\beta(t-s)} (e^{-\beta h} + 1)) \in \mathbb{L}^1([0, T], ds),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h \rightarrow 0} \mathbb{E}|I_{51}(h)|^2 = 0.$$

Again by Lemma 4, we get that

$$\mathbb{E}|I_{52}(h)|^2 \leq \frac{2Ht^{2H-1}LM^2(1 - e^{-2\beta h})}{2\beta}.$$

Thus,

$$\lim_{h \rightarrow 0} \mathbb{E}|I_{52}(h)|^2 = 0.$$

The above arguments show that  $\lim_{h \rightarrow 0} \mathbb{E} \|\psi(x)(t+h) - \psi(x)(t)\|^2 = 0$ . Hence, we conclude that the function  $t \rightarrow \psi(x)(t)$  is continuous on  $[0, T]$  in the  $\mathbb{L}^2$ -sense.

**Step 2:** Now, we are going to show that  $\psi$  is a contraction mapping in  $S_{T_1}(\varphi)$  with some  $T_1 \leq T$  to be specified later. Let  $x, y \in S_T(\varphi)$ , by using the

inequality  $(a+b+c)^2 \leq \frac{1}{\nu}a^2 + \frac{2}{1-\nu}b^2 + \frac{2}{1-\nu}c^2$ , where  $\nu := L_*M_* < 1$ , we obtain for any fixed  $t \in [0, T]$

$$\begin{aligned}
\|\psi(x)(t) - \psi(y)(t)\|^2 &\leq \frac{1}{\nu} \|g(t, x(t-r(t))) - g(t, y(t-r(t)))\|^2 \\
&\quad + \frac{2}{1-\nu} \left\| \int_0^t U(t, s) A(s) (g(s, x(s-r(s))) - g(s, y(s-r(s)))) ds \right\|^2 \\
&\quad + \frac{2}{1-\nu} \left\| \int_0^t U(t, s) (f(s, x(s-\rho(s))) - f(s, y(s-\rho(s)))) ds \right\|^2 \\
&= \sum_{k=1}^3 J_k(t).
\end{aligned}$$

By using the fact that the operator  $\|(A^{-1}(t))\|$  is bounded, combined with the condition  $(\mathcal{H}.3)$ , we obtain that

$$\begin{aligned}
\mathbb{E}\|J_1(t)\| &\leq \|A^{-1}(t)\|^2 \mathbb{E}\|A(t)g(t, x(t-r(t))) - A(t)g(t, y(t-r(t)))\| \\
&\leq \frac{L_*^2 M_*^2}{\nu} \mathbb{E}\|x(t-r(t)) - y(t-r(t))\|^2 \\
&\leq \nu \sup_{s \in [-\tau, t]} \mathbb{E}\|x(s) - y(s)\|^2.
\end{aligned}$$

By hypothesis  $(\mathcal{H}.3)$  combined with Hölder's inequality, we get that

$$\begin{aligned}
\mathbb{E}\|J_2(t)\| &\leq \mathbb{E}\left\| \int_0^t U(t, s) [A(t)g(t, x(t-r(t))) - A(t)g(t, y(t-r(t)))] ds \right\| \\
&\leq \frac{2}{1-\nu} \int_0^t M^2 e^{-2\beta(t-s)} ds \int_0^t \mathbb{E}\|x(s-r) - y(s-r)\|^2 ds \\
&\leq \frac{2M^2 L_*^2}{1-\nu} \frac{1-e^{-2\beta t}}{2\beta} t \sup_{s \in [-\tau, t]} \mathbb{E}\|x(s) - y(s)\|^2.
\end{aligned}$$

Moreover, by hypothesis  $(\mathcal{H}.2)$  combined with Hölder's inequality, we can conclude that

$$\begin{aligned}
E\|J_3(t)\| &\leq E\left\| \int_0^t U(t, s) [f(s, x(s-\rho(s))) - f(s, y(s-\rho(s)))] ds \right\| \\
&\leq \frac{2C_1^2}{1-\nu} \int_0^t M^2 e^{-2\beta(t-s)} ds \int_0^t \mathbb{E}\|x(s-r) - y(s-r)\|^2 ds
\end{aligned}$$

$$\leq \frac{2M^2 C_1^2}{1-\nu} \frac{1-e^{-2\beta t}}{2\beta} t \sup_{s \in [-\tau, t]} \mathbb{E} \|x(s) - y(s)\|^2.$$

Hence

$$\sup_{s \in [-\tau, t]} \mathbb{E} \|\psi(x)(s) - \psi(y)(s)\|^2 \leq \gamma(t) \sup_{s \in [-\tau, t]} \mathbb{E} \|x(s) - y(s)\|^2,$$

where

$$\gamma(t) = \nu + [L_*^2 + C_1^2] \frac{2M^2}{1-\nu} \frac{1-e^{-2\beta t}}{2\beta} t$$

By condition  $(\mathcal{H}.3)$ , we have  $\gamma(0) = \nu = L_* M_* < 1$ . Then there exists  $0 < T_1 \leq T$  such that  $0 < \gamma(T_1) < 1$  and  $\psi$  is a contraction mapping on  $S_{T_1}(\varphi)$  and therefore has a unique fixed point, which is a mild solution of equation (1) on  $[-\tau, T_1]$ . This procedure can be repeated in order to extend the solution to the entire interval  $[-\tau, T]$  in finitely many steps. This completes the proof. ■

#### 4. An Example

Let us consider the following stochastic partial neutral functional differential equation with finite variable delays driven by a cylindrical fractional Brownian motion:

$$\begin{cases} d[u(t, \zeta) + G(t, u(t-r(t), \zeta))] &= \left[ \frac{\partial^2}{\partial^2 \zeta} u(t, \zeta) + b(t, \zeta) u(t, \zeta) + F(t, u(t-\rho(t), \zeta)) \right] dt \\ &+ \sigma(t) dB^H(t) \\ u(t, 0) + G(t, u(t-r(t), 0)) &= 0, \quad t \geq 0 \\ u(t, \pi) + G(t, u(t-r(t), \pi)) &= 0, \quad t \geq 0 \\ u(t, \zeta) = \varphi(t, \zeta), &t \in [-\tau, 0] \quad 0 \leq \zeta \leq \pi, \end{cases} \quad (9)$$

where  $B^H$  is a fractional Brownian motion,  $b(t, \zeta)$  is a continuous function and is uniformly Hölder continuous in  $t$ ,  $F, G : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$  are continuous functions. To study this system, we consider the space  $X = \mathbb{L}^2([0, \pi], \mathbb{R})$  and the operator  $A : D(A) \subset X \longrightarrow X$  given by  $Ay = y''$  with

$$D(A) = \{y \in X : y'' \in X, \quad y(0) = y(\pi) = 0\}.$$



It is well known that  $A$  is the infinitesimal generator of an analytic semi-group  $\{T(t)\}_{t \geq 0}$  on  $X$ . Furthermore,  $A$  has discrete spectrum with eigenvalues  $-n^2$ ,  $n \in \mathbb{N}$  and the corresponding normalized eigenfunctions given by

$$e_n := \sqrt{\frac{2}{\pi}} \sin nx, \quad n = 1, 2, \dots$$

In addition  $(e_n)_{n \in \mathbb{N}}$  is a complete orthonormal basis in  $X$  and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n$$

for  $x \in X$  and  $t \geq 0$ . It follows from this representation that  $T(t)$  is compact for every  $t > 0$  and that  $\|T(t)\| \leq e^{-t}$  for every  $t \geq 0$ .

On the domain  $D(A)$ , we define the operators  $A(t) : D(A) \subset X \longrightarrow X$  by

$$A(t)x(\zeta) = Ax(\zeta) + b(t, \zeta)x(\zeta).$$

By assuming that  $b(., .)$  is continuous and that  $b(t, \zeta) \leq -\gamma$  ( $\gamma > 0$ ) for every  $t \in \mathbb{R}$ ,  $\zeta \in [0, \pi]$ , it follows that the system

$$\begin{cases} u'(t) &= A(t)u(t), \quad t \geq s, \\ u(s) &= x \in X, \end{cases}$$

has an associated evolution family given by

$$U(t, s)x(\zeta) = \left[ T(t-s) \exp\left(\int_s^t b(\tau, \zeta) d\tau\right) x \right](\zeta).$$

From this expression, it follows that  $U(t, s)$  is a compact linear operator and that for every  $s, t \in [0, T]$  with  $t > s$

$$\|U(t, s)\| \leq e^{-(\gamma+1)(t-s)}$$

In addition,  $A(t)$  satisfies the assumption  $\mathcal{H}_1 - ii$ ) (see Aouad and Baghli (2013)).

In order to define the operator  $Q : Y := L^2([0, \pi], \mathbb{R}) \longrightarrow Y$ , we choose a sequence  $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$ , set  $Qe_n = \lambda_n e_n$ , and assume that

$$\text{tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$

Define the fractional Brownian motion in  $Y$  by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta^H(t) e_n,$$

where  $H \in (\frac{1}{2}, 1)$  and  $\{\beta_n^H\}_{n \in \mathbb{N}}$  is a sequence of one-dimensional fractional Brownian motions mutually independent.

To write the initial-boundary value problem (9) in the abstract form we assume the following: Letting  $u(t)(\cdot) = u(t, \cdot)$ . For  $t \in [0, T]$ ,  $u \in X$  and  $\zeta \in [0, \pi]$

- i) The substitution operator  $f : [0, T] \times X \longrightarrow X$  defined by  $f(t, u)(\cdot) = F(t, u(\cdot))$  is continuous and we impose suitable conditions on  $F$  to verify assumption  $\mathcal{H}_2$ .
- ii) The substitution operator  $g : [0, T] \times X \longrightarrow X$  defined by  $g(t, u)(\cdot) = G(t, u(\cdot))$  is continuous and we impose suitable conditions on  $G$  to verify assumptions  $\mathcal{H}_2$  and  $\mathcal{H}_3$ .
- iii) The function  $\sigma : [0, T] \longrightarrow \mathcal{L}_2^0(L^2([0, \pi], \mathbb{R}), L^2([0, \pi], \mathbb{R}))$  is bounded, that is, there exists a positive constant  $L$  such that  $\|\sigma(t)\|_{\mathcal{L}_2^0} \leq L$  uniformly in  $t \in [0, T]$ .

Thus the problem (9) can be written in the abstract form

$$\begin{cases} d[x(t) + g(t, x(t - r(t)))] = [A(t)x(t) + f(t, x(t - \rho(t)))]dt + \sigma(t)dB^H(t), & 0 \leq t \leq T, \\ x(t) = \varphi(t), & -\tau \leq t \leq 0. \end{cases}$$

Furthermore, if we impose suitable condition on the delay functions  $r(\cdot)$ ,  $\rho(\cdot)$  and on the initial value  $\varphi$  to verify assumptions on theorem 8, we can conclude that the system 9 has a unique mild solution on  $[-\tau, T]$ .

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